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Free σ-Products and Noncommutatively Slender Groups

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An infinitary version of the notion of free products has been introduced and investigated by G. Higman [11]. Let G_i $(i \in I)$ be groups and $*_{i \in X} G_i$ the free product of G_i $(i \in X)$ for $X \subset I$ and $p_{XY}: *_{i \in Y} G_i \to *_{i \in X} G_i$ the canonical homomorphism for $X \subset Y \subset I$. Then, the unrestricted free product is the inverse limit $\lim_{\leftarrow}(*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I)$, where $Y \Subset I$ means that Y is a finite subset of I. In the present paper we introduce a similar one to the unrestricted free product, which is a subgroup of the unrestricted free product and equal to the subgroup P in [11, Section 6] if $G_i \simeq \mathbb{Z}$ and I is countable. There were also related investigations due to H. B. Griffiths [8, 9]. Free products are defined using words of finite length. Our infinitary version of free products will be defined using words of infinite length instead of finite one. The group $\times_{i \in I} G_i$ is called a free complete product and is isomorphic to a subgroup of the unrestricted free product, that is, $\bigcap_{F \Subset I} \{*_{i \in F} G_i * \lim_{\leftarrow} (*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I \backslash F)\}$. Our interest will be concentrated to free σ-products, which are defined using words of countable length and a subgroup of the free complete product. One reason to do so is that free σ-products are naturally related to fundamental groups of certain spaces [9], as we shall explain and state applications in the Appendix. Another reason is that these behave well concerning noncommutatively slender groups, which will be defined later, but we have not found a slender property concerning free complete products.

In Section 1 we define free complete products and free σ-products and state some preliminary results. In Section 2 we prove a noncommutative version of Chase's lemma, that is, a theorem about homomorphisms from free σ-products to free products of infinite components. In Section 3 we introduce a new notion "noncommutatively slender groups" and investigate it. We remark that this notion is strictly stronger than that of slender

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groups in the sense of [7]. In Section 4 we investigate the abelianizations of free σ-products and related ones. In the Appendix we explain the relationship with algebraic topology.

First, we state basic notations. For a subset X of a group G, $\langle X \rangle$ is the subgroup generated by X. The direct product $\prod_{i \in I} G_i$ is the group consisting of all functions x from the index set I such that $x(i) \in G_i$ $(i \in I)$. The restricted direct product $\prod^r_{i \in I} G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of all x such that $\{i : x(i) \neq e\}$ is finite. (The symbol "e" is always used for the identity of a group in question. We use "0" instead of "e" for an abelian group as usual.) The σ-product $\prod^\sigma_{i \in I} G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of all x such that $\{i : x(i) \neq e\}$ is countable. In case G_i $(i \in I)$ are isomorphic to a group G, $\prod_{i \in I} G_i$ is denoted by $\prod_I G$. The group of rational integers is denoted by $\mathbb{Z}$ and the set of natural numbers is denoted by N.

1. Free Complete Products and Free σ-products

First we introduce words of infinite length.

Definition 1.1. Let G_i $(i \in I)$ be groups. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. Elements of $\bigcup_{i \in I} G_i$ are called letters. W is a word, if W is a function from a linearly ordered set $\bar{W}$ to $\bigcup_{i \in I} G_i$ such that $W^{-1}(G_i)$ is finite for each i. In case the cardinality of $\bar{W}$ is countable, we say that W is a σ-word. The class of all words is denoted by $\mathscr{W}(G_i : i \in I)$ (abbreviated by $\mathscr{W}$) and the class of all σ-words is denoted by $\mathscr{W}^\sigma(G_i : i \in I)$ (abbreviated by $\mathscr{W}^\sigma$).

If there exists an isomorphism $i: \bar{U} \to \bar{V}$ as linearly ordered sets and $U(\alpha) = V(i(\alpha))$ for all $\alpha \in \bar{U}$, we say that U and V are isomorphic and denote it by $U \simeq V$. In this case we identify U and V. Since the cardinality of $\bar{W}$ is less than or equal to $\mathrm{Max}\{|I|, \aleph_0\}$ for a word W, $\mathscr{W}$ becomes a set under this identification. For words of finite length, this is the same as the usual definition. For the definition of free products we refer the reader to [10 or 13]. For a word $W \in \mathscr{W}(G_i : i \in I)$ and a subset $X \subset I$, W_X is the word obtained by eliminating letters not in $\bigcup_{i \in X} G_i$; that is, $W_X \in \mathscr{W}(G_i : i \in X)$, $\overline{W_X} = \{\alpha \in \bar{W} : W(\alpha) \in \bigcup_{i \in X} G_i\}$, and $W_X(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_X}$. For words U and V, we say that $U \sim V$ holds if $U_F = V_F$ for every $F \Subset I$, where we regard U_F, V_F as elements of the free product $*_{i \in F} G_i$. Then, $\sim$ is an equivalence relation on $\mathscr{W}$ clearly. Denote the equivalence class containing U by $[U]$. For $U, V \in \mathscr{W}$, let UV be the composition of U and V, that is, $\overline{UV} = \{(0, \alpha), (1, \beta) : \alpha \in \bar{U}, \beta \in \bar{V}\}$, where $(0, \alpha) < (1, \beta)$ for $\alpha \in \bar{U}$ and $\beta \in \bar{V}$ and $(i, \alpha) < (i, \beta)$ for $\alpha < \beta$ and $i = 0, 1$; $UV((0, \alpha)) = U(\alpha)$ and $UV((1, \beta)) = V(\beta)$. Let U^{-1} be the word such that $\overline{U^{-1}} =$

$\{(0, \alpha) : \alpha \in \bar{U}\}$, where $(0, \alpha) < (0, \beta)$ if $\alpha > B$ and $U^{-1}((0, \alpha)) = U(\alpha)^{-1}$. Then, $\mathscr{W}/\sim = \{[W] : W \in \mathscr{W}\}$ clearly becomes a group with its operation $[U][V] = [UV]$. We define U^0 as the empty word, $U^{n+1} = U^n U$ and $U^{-n-1} = U^{-n}U^{-1}$ for $n \in N$.

DEFINITION 1.2. The free complete product $\times_{i \in I} G_i$ is the group $\mathscr{W}(G_i : i \in I)/\sim$. The free σ-product $\times^{\sigma}_{i \in I} G_i$ is the group $\mathscr{W}^{\sigma}(G_i : i \in I)/\sim$, which is a subgroup of $\times_{i \in I} G_i$. In case every G_i is isomorphic to G, we abbreviate $\times_{i \in I} G_i$ by $\times_I G$ and similarly for free σ-products.

Restricting the length of words to be finite, we obtain the free product $*_{i \in I} G_i$. Obviously, $\times_{i \in I} G_i$ and $\times^{\sigma}_{i \in I} G_i$ are isomorphic to $*_{i \in I} G_i$, if I is finite. We define reduced words and shall show that every word corresponds to a unique reduced word.

DEFINITION 1.3. A word W is reduced, if $W \simeq UXV$ implies $[X] \neq e$ for any non-empty word X, where e is the identity, and for any neighboring elements α and β of $\bar{W}$ it never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same G_i. A word W is quasi-reduced, if $W \simeq UXV$ with $[X] = e$ implies $\mathrm{Im}(X) \subset G_i$ and the existence of $e \neq g \in G_i$ for some i such that g is the rightmost letter of U or the leftmost letter of V.

In other words, W is quasi-reduced if a reduced word is obtained by multiplying all neighboring letters which belong to the same G_i.

THEOREM 1.4. *For any word W, there exists a reduced word V such that $[W] = [V]$ and V is unique up to isomorphism.*

Proof. We define words W_μ for ordinals μ by induction. Let W_0 be W. If there exists a non-empty word X such that W_μ is isomorphic to UXV and $[X] = e$, let $\overline{W_{\mu+1}} = \{\alpha \in \overline{W_\mu} : i(\alpha) \in \bar{U}$ or $i(\alpha) \in \bar{V}\} \subset \bar{W}$ and $W_{\mu+1}(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_{\mu+1}}$, where the ordering is the restriction of that of $\bar{W}$ and $i : \overline{W_\mu} \to \overline{UXV}$ is the order isomorphism. Otherwise, the procedure is completed. For a limit ordinal μ, let $\overline{W_\mu} = \bigcap_{\nu > \mu} \overline{W_\nu}$ and $W_\mu(\alpha) = W(\alpha)$ for $\alpha \in \bar{W}_\mu$. This procedure must stop at some ordinal whose cardinality is at most $\mathrm{Max}\{|I|, \aleph_0\}$ because the cardinality of $\bar{W}$ is equal to or less than $\mathrm{Max}\{|I|, \aleph_0\}$. Let W_∞ be the obtained word. By induction we can see that $[W_\mu] = [W]$ and hence $[W_\infty] = [W]$. There may be a neighboring $\alpha, \beta \in \overline{W_\infty}$ such that $W_\infty(\alpha)$ and $W_\infty(\beta)$ belong the same G_i. Since such occasions happen only finitely many times for each i, performing the calculation in each G_i we obtain the desired reduced word of W. Next, suppose that $[U] = [V]$ for reduced words U and V. We define $\varphi : \bar{U} \to \bar{V}$ in the following manner. For $\alpha \in \bar{U}$ there exists a unique $i \in I$ such that $U(\alpha) \in G_i$. Then, there exist $i \in E \Subset I$, letters $g_1, \ldots, g_m \in G_i$ and

$X_1 \cdots X_{m+1} \in \mathscr{W}(G_j: i \neq j \in I)$ such that $U \simeq X_1 g_1 X_2 \cdots X_m g_m X_{m+1}$, α corresponds to g_k, $U_E \simeq (X_1)_E g_1 (X_2)_E \cdots (X_m)_E g_m (X_{m+1})_E$, and $[(X_2)_E] \cdots [(X_m)_E] \neq e$. On the other hand, there exist $E \subset F \Subset I$, letters $g'_1 \cdots g'_n \in G_i$ and $Y_1 \cdots Y_{n+1} \in \mathscr{W}(G_j: i \neq j \in I)$ such that $V \simeq Y_1 g'_1 Y_2 \cdots Y_n g'_n Y_{n+1}$, $V_F \simeq (Y_1)_F g'_1 (Y_2)_F \cdots (Y_n)_F g'_n (Y_{n+1})_F$, and $[(Y_2)_F] \cdots [(Y_n)_F] \neq e$.

Since $[U_F] = [V_F]$ by definition, $m = n$ and $g(l) = g'(l)$ for $1 \leqslant l \leqslant m$. Let $\varphi(\alpha) \in \bar{V}$ be the member corresponding to g'_k in V. Clearly φ is a 1–1 onto map and $U(\alpha) = V(\varphi(\alpha))$. Taking large enough $F \Subset I$ as the above, we can see that φ preserves the order. Therefore, U and V are isomorphic.

From now on we regard a word as an element of $\times_{i \in I} G_i$ so that no confusion will occur. Hence, $U = V$ means $[U] = [V]$ for words U and V.

COROLLARY 1.5. *Let U and V be reduced words. If $UV = e$, then V is isomorphic to U^{-1}.*

COROLLARY 1.6. *Let U be a reduced word. There exists no nonempty reduced word X such that $U = UX$ or $U = XU$. If U is nonempty and $U = U^{-1}$, then there exist a reduced word X and a letter g such that U is isomorphic to $X^{-1}gX$ and $g^2 = e$.*

Proof. The first proposition is clear. Since U^{-1} is also reduced, $U = U^{-1}$ implies $U \simeq U^{-1}$ and, hence, let $i: \bar{U} \to \overline{U^{-1}}$ be the order isomorphism. Under the notation before Definition 1.2, let $\bar{X}$ be the maximal subset of $\bar{U}$ such that $\alpha > \beta \in \bar{X}$ implies $\alpha \in \bar{X}$ and $i^{-1}(0, \alpha) \notin \bar{X}$ for any $\alpha \in \bar{X}$, and let $X(\alpha) = U(\alpha)$ for $\alpha \in \bar{X}$. If $\bar{X} \cup i^{-1}\{(0, \alpha) : \alpha \in \bar{X}\} = \bar{U}$ then $U = e$. Hence $U \simeq X^{-1}gX$ for some letter $g \neq e$ by the maximality of $\bar{X}$. Then, $g^2 = e$ by $U \simeq U^{-1}$.

Considering the reduction in the proof of Theorem 1.4, we obtain

COROLLARY 1.7. *Let U and V be reduced words. Then, there exist reduced words X, Y, Z such that $U \simeq XY$, $V \simeq Y^{-1}Z$, and XZ is quasi-reduced.*

Next we show another presentation of $\times_{i \in I} G_i$ as a subgroup of an inverse limit.

PROPOSITION 1.8. *The free complete product $\times_{i \in I} G_i$ is isomorphic to $\bigcap_{F \Subset I} *_{i \in F} G_i * \lim_{\leftarrow}(*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I \backslash F)$, which is a subgroup of $\lim_{\leftarrow}(*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I)$.*

Proof. Define $\varphi_X: \times_{i \in I} G_i \to *_{i \in X} G_i$ for $X \Subset I$ as $\varphi_X(W) = W_X$ for a word W. Then, φ_X is a homomorphism by definition and $p_{XY} \cdot \varphi_Y = \varphi_X$

for $X \subset Y \Subset I$. Let φ: $\times_{i \in I} G_i \to \lim_{\leftarrow}(*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I)$ be the homomorphism induced by φ_X $(X \Subset I)$. Then, φ is clearly injective. Let $x \in \bigcup_{F \Subset I} *_{i \in F} G_i * \lim_{\leftarrow}(*_{i \in X} G_i, p_{XY}: X \subset Y \Subset I \backslash F)$. For each $i \in I$, let W_i be the reduced word corresponding to x as a member of $G_i * \lim_{\leftarrow}(*_{j \in X} G_j, p_{XY}: X \subset Y \Subset I \backslash \{i\})$. Let $g(i, 1) \cdots g(i, k_i)$ be the sequence of letters in G_i appearing in W_i in this order. Now let $\bar{W} = \{(i, 1) \cdots (i, k_i) : i \in I\}$. Consider the reduced word $W_{i,j}$ corresponding to x as a member of $G_i * G_j * \lim_{\leftarrow}(*_{k \in X} G_k, p_{XY}: X \subset Y \Subset I \backslash \{i, j\})$; then we can see that $g(i, 1) \cdots g(i, k_i)$ and $g(j, 1) \cdots g(i, k_j)$ are appearing in $W_{i,j}$. Define $(i, p) < (j, q)$ if $g(i, p)$ is left of $g(j, q)$ in the word $W_{i,j}$. Then, it is easy to see that this is a linear ordering on $\bar{W}$. Let $W(i, p) = g(i, p)$, then $W \in \mathscr{W}(G_i: i \in I)$ and $\varphi(W) = x$.

For $x \in \times_{i \in I} G_i$, the i-length of x (say $l_i(x)$) is the cardinality of $\{\alpha \in \bar{W} : W(\alpha) \in G_i\}$, where W is the reduced word of x. $\times_{i \in I} G_i$ and $\times^{\sigma}_{i \in I} G_i$ naturally admit infinite operations for certain sequences as $\prod_{i \in I} G_i$. Namely,

Proposition 1.9. *Let $g_\lambda (\lambda \in \Lambda)$ be elements of $\times_{i \in I} G_i$ such that $\{\lambda \in \Lambda : l_i(g_\lambda) \neq e\}$ are finite for all $i \in I$ and denote the element corresponding 1 of the λth component of $\times_\Lambda \mathbb{Z}$ by δ_λ. Then, there exists a natural homomorphism φ: $\times_\Lambda \mathbb{Z} \to \times_{i \in I} G_i$ such that $\varphi(\delta_\lambda) = g_\lambda$ $(\lambda \in \Lambda)$. Consequently, in case $\Lambda = N$ and $g_n \in \times^{\sigma}_{i \in I} G_i (n \in N)$, we obtain φ: $\times_N \mathbb{Z} \to \times^{\sigma}_{i \in I} G_i$ so that $\varphi(\delta_n) = g_n$ $(n \in N)$.*

Proof. Let W_λ be the reduced word of g_λ for $\lambda \in \Lambda$. For $W \in \mathscr{W}(\mathbb{Z}, \lambda \in \Lambda)$, let $\overline{W^*} = \{(\alpha, \beta) : \alpha \in \bar{W},\ \beta \in \overline{W^a_\lambda}$, where $W(\alpha) = a\delta_\lambda$ for $a \in \mathbb{Z}\}$ and $(\alpha, \beta) < (\alpha', \beta')$ if and only if $\alpha < \alpha'$, or $\alpha = \alpha'$ and $\beta < \beta'$. And let $W^*((\alpha, \beta)) = W^a_\lambda(\beta)$, where $W(\alpha) = a\delta_\lambda$. Finally, let $\varphi(W) = W^*$. It is easy to check by Corollary 1.5 that φ is the desired homomorphism.

2. A Noncommutative Version of Chase's Lemma

Roughly speaking, Chase's lemma [1] says any that homomorphism from an infinite direct product to an infinite direct sum maps a large part to a small part. More precisely, let $h: \prod_{n \in N} A_n \to \bigoplus_{j \in J} B_j$ be a homomorphism for abelian groups A_n $(n \in N)$ and B_j $(j \in J)$. Then, there exist $k, m \in N$ and a finite subset F of J such that $h(m \cdot \prod_{n \geqslant k} A_n) \leqslant \bigoplus_{j \in F} B_j + U(\bigoplus_{j \in J} B_j)$, where $U(X)$ is the Ulm subgroup of X, that is, $\bigcap_{n \in N} nX$. We prove the following:

Theorem 2.1. (A noncommutative version of Chase's lemma). *Let h: $\times^{\sigma}_{i \in I} G_i \to *_{j \in J} H_j$ be a homomorphism for groups G_i $(i \in I)$ and H_j $(j \in J)$. Then, there exist $E \Subset I$ and $F \Subset J$ such that $h(\times^{\sigma}_{i \in I \backslash E} G_i) \leqslant *_{j \in F} H_j$.*

To show this some notion and lemmas are necessary. For $g \in *_{j \in J} H_j$, $l(g)$ denotes the length of the reduced word corresponding to g. Let $W \simeq Xg$, where W, X are words and g is a letter. We say that g is stable in W, if the reduced word corresponding to Xg is of form Ug. Similarly for $W \simeq gX$.

LEMMA 2.2. *Let H_j $(j \in J)$ be groups and U and X be reduced words. If the leftmost letter g or the rightmost one g^{-1} in XUX^{-1} is not stable in XUX^{-1}, then $l(XUX^{-1}) \leqslant l(U) + 1$.*

Proof. It is enough to deal with the case that g^{-1} is not stable. Let V be the reduced word of XU. Then, $l(V) \leqslant l(X) + l(U)$. Since the rightmost letter g^{-1} in VX^{-1} is not stable, $l(VX^{-1}) \leqslant l(V) - l(X^{-1}) + 1$. Therefore, $l(XUX^{-1}) \leqslant l(X) + 1$.

LEMMA 2.3. *Let H_j $(j \in J)$ be groups. Let $m + n + 2 \leqslant k$ for $m, n, k \in N$ and $u, x_i, z \in *_{j \in J} H_j$ $(1 \leqslant i \leqslant M)$. If $l(u) \leqslant m$, $u = x_1 z^k \cdots x_M z^k$ and $l(x_i) \leqslant n$ for all $1 \leqslant i \leqslant M$, then z is a conjugate of a member of some H_j or $z = x^{-1} fxy^{-1} gy$ for some $f \in H_j$ and $g \in H_{j'}$ with $f^2 = g^2 = e$.*

Proof. It is easy to see that there exist reduced words U and W such that $z = W^{-1} UW$ and that both words UU and $X^{-1} UW$ are quasi-reduced. If $l(U) \geqslant 2$, we can take the above U and W so that UU is reduced. If $l(U) \leqslant 1$, the proof is done. Hence, we assume $l(U) \geqslant 2$ and also assume that UU is reduced. Let X_i be the reduced word of x_i for each $1 \leqslant i \leqslant M$. Then, $x_1 y_1 \cdots x_M y_M = X_1 W^{-1} U^k W X_2 W^{-1} \cdots W X_M W^{-1} U^k W$. Suppose that the leftmost letter g and the rightmost one g^{-1} of $WX_i W^{-1}$ are stable in $WX_i W^{-1}$. Then, the reduced word of $U^2 W X_i W^{-1} U^2$ is of form $UY_i U$. On the other hand, if at least one of g and g^{-1} is not stable, then $l(WX_i W^{-1}) \leqslant l(X_i) + 1$ by Lemma 2.2. Let Z_i be the reduced word of $WX_i W^{-1}$. Let p be the least number so that $2p \geqslant n + 1$. Then, the reduced word of $Z_i U^{p+1}$ is of form $Y_i U$, where $Y_i \simeq Y_i' V_i$ and $l(Y_i') \leqslant l(Z_i)$ and $U \simeq W_i V_i$ for some W_i. Hence, the reduced word of $U^{p+2} Y_i$ is of form UZ_i'. Suppose that the reduced word of $UZ_i' U^2$ is of form $UZ_i'' U$. Since $l(U^k) \geqslant l(U) + 2(k-1)$, the reduced form of $X_1 W^{-1} U^k W X_2 W^{-1} U^k \cdots X_M W^{-1} U^k W$ is of form $P_1 U P_2 U \cdots P_M U^m V$, where $V = UW$. This contradicts $l(u) \leqslant m$. Therefore, the reduced word of $UZ_i' U^2$ is not of form $UZ_i'' U$. Since UU is reduced, not only the rightmost letter of Y_i is not stable in $U^{p+2} Y_i$, but also Z_i' must be of form $U^d S_i$ for some d and $U \simeq S_i T_i$ for some T_i. By the assumption, S_i must disappear in the reduction of $UU^d S_i U^2$ and hence $S_i \simeq S_i^{-1}$. By Corollary 1.6, S_i is empty or $S_i = x^{-1} fx$ for some $f \in H_j$ with $f^2 = e$. If S_i is empty, then $UZ_i U^2$ itself is reduced, which is a contradiction. Hence, the latter holds. Since UU is reduced and $l(U) \geqslant 2$, T_i is not empty. Apply the same reasoning for S_i to T_i, then we obtain that $T_i = y^{-1}gy$ for some $g \in H_{j'}$ with $g^2 = e$.

LEMMA 2.4. *Let $h\colon \times_N \mathbb{Z} \to *_{j \in J} H_j$ be a homomorphism. Then, there exists $F \Subset J$ such that $h(\times_N \mathbb{Z}) \leqslant *_{j \in F} H_j$.*

Proof. By Kuro's theorem [13, Section 34 or 10, Chap. 17], $h(\times_N \mathbb{Z})$ is a free product of copies of $\mathbb{Z}$ and conjugate groups of subgroups of some H_j. If the number of components of this free product is finite, then we obtain the conclusion. Hence, it suffices to deduce a contradiction from $h(\times_N \mathbb{Z}) = *_{j \in J} H_j$ for infinite J. Let $p_j\colon *_{j \in J} H_j \to H_j$ be the projection. First, we inductively define $k_n \in N$, $j_n \in J$, $x_n \in \times_{N \setminus \{1 \cdots n\}} \mathbb{Z}$ and finite subsets F_n of J for $n = 0, 1 \cdots$. Let $k_0 = 1$ and take x_0 and finite $F_0 \subset J$ so that $h(x_0) \in *_{j \in F_0} H_j$, $p_{j_\alpha} \cdot h(x_0) \neq e$ for $0 \leqslant \alpha \leqslant 4$, where the j'_αs are distinct. Suppose that we have defined the $(n-1)$-step. Since $h(\times_{\{1 \cdots n\}} \mathbb{Z} \leqslant *_{j \in E} H_j$ for some finite E and h is surjective, there exist $x_n \in \times_{N \setminus \{1 \cdots n\}} \mathbb{Z}$ and distinct $j_{5n+\alpha} \in J \setminus F_n$ such that $p_{j_{5n+\alpha}} \cdot h(x_n) \neq e$ for $0 \leqslant \alpha \leqslant 4$. Let $k_n = n + 2 + \max\{l(h(x_k \cdots x_{n-1})) : 0 \leqslant k \leqslant n-1\}$ and F_n be a finite subset of J such that $F_{n-1} \subset F_n$ and $h(x_n) \in *_{j \in F_n} H_j$. Then, let *Seq* be the set of all finite sequences of natural numbers and denote the length of $s \in \mathrm{Seq}$ by $lh(s)$. The empty sequence is denoted by $\langle\ \rangle$ and generally $s \in \mathrm{Seq}$ is denoted by $\langle s_1 \cdots s_n \rangle$, where $s_k \in N$ $(1 \leqslant k \leqslant n)$. For $s, t \in \mathrm{Seq}$, $s \prec t$ if $s(i) < t(i)$ for the minimal i with $s(i) \neq t(i)$ or t extends s. Let $D_n = \{s \in \mathrm{Seq} : 0 \leqslant lh(s) \leqslant n,\ 1 \leqslant s(i) \leqslant k_i \text{ for } 1 \leqslant i \leqslant n\}$ and $\overline{W_n} = D_n$ with the ordering $\prec$ and $W_n(s) = x_n$, where $n = lh(s)$. Similarly, let $D_n^{(m)} = \{s \in \mathrm{Seq} : 0 \leqslant lh(s) \leqslant n,\ 1 \leqslant s(i) \leqslant k_{m+i} \text{ for } 1 \leqslant i \leqslant n\}$ and $W_n^{(m)} = D_n^{(m)}$ with the ordering $\prec$ and $W_n^{(m)}(s) = x_{m+n}$ for $n = lh(s)$. Then, there exist σ-words W and $W^{(m)}$ $(m \in N)$ such that $W_{\{1 \cdots n\}} = (W_{n-1})_{\{1 \cdots n\}}$ and $(W^{(m)})_{\{1 \cdots n\}} = (W_{n-1}^{(m)})_{\{1 \cdots n\}}$ for $n \in N$. There exists $E \Subset J$ such that $h(W) \in *_{j \in E} H_j$. Let m be a number such that $E \cap F_{m-1} = E \cap \bigcup_{n \in N} F_n$ and $l(h(W)) \leqslant m$. Then, $h(x_k) \in *_{j \in F_{m-1}} H_j$ for $0 \leqslant k \leqslant m-1$. Since $h(W) = y_1 \cdot h(W^{(m)})^{k_m} \cdot y_2 \cdot h(W^{(m)})^{k_m} \cdots y_M \cdot h(W^{(m)})^{k_m}$ for some y_k with $l(y_k) \leqslant k_m - (m+2)$, $p_{j_{m+\alpha}} \cdot h(W^{(m)}) = e$ for at least three $\alpha \in \{0, 1, 2, 3, 4\}$ by Lemma 2.3. A similar argument for $h(W^{(m+1)})^{k_{m+1}}$ and the fact $W^{(m)} = x_m \cdot (W^{(m+1)})^{k_{m+1}}$ imply that $p_{j_{m+\alpha}}(x_m) = e$ for at least one $\alpha \in \{0, 1, 2, 3, 4\}$, which is a contradiction.

Proof of Theorem 2.1. Suppose the negation of the conclusion. Then, we obtain $x_n \in \times^\sigma_{i \in I \setminus E_n} G_i$, $E_n \Subset I$, and $F_n \Subset J$, such that $E_n \subset E_{n+1}$, $F_n \subset F_{n+1}$, $x_n \in \times_{U_n E_n} \mathbb{Z}$, $h(x_n) \notin *_{j \in F_n} H_j$ and $h(x_n) \in *_{j \in F_{n+1}} H_j$ $(n \in N)$. Finally, we obtain a contradiction by Lemma 2.4 and Proposition 1.9.

COROLLARY 2.5. *Let $h\colon \times^\sigma_{i \in I} G_i \to *_{j \in J} H_j$ be a homomorphism for groups G_i $(i \in I)$ and H_j $(j \in J)$. If every G_i is finitely generated, then there exist $F \Subset J$ such that $h(\times^\sigma_{i \in I} G_i) \leqslant *_{j \in F} H_j$.*

We remark that Theorem 2.1 for the unrestricted free product can be shown similarly, if an index set I is countable.

3. Noncommutatively Slender Groups

We introduce a new notion "noncommutatively slender groups," which is a noncommutative version of slender abelian groups [6, Section 94]. Recall that an abelian group A is slender if and only if for any homomorphism $h: \prod_N \mathbb{Z} \to A$ there exists $n \in N$ such that $h(\prod_{N\setminus\{1 \cdots n\}}\mathbb{Z}) = \{0\}$.

Definition 3.1. A group G is noncommutatively slender, if for any homomorphism $h: \times_N \mathbb{Z} \to G$ there exists an $n \in N$ such that $h(\times_{N\setminus\{1 \cdots n\}} \mathbb{Z}) = \{e\}$. We say "$n$-slender" instead of "noncommutatively slender" for short.

This notion is equivalent to a seemingly weaker condition as in case of slender abelian groups, which we show now.

Proposition 3.2. *If for any homomorphism $h: \times_N \mathbb{Z} \to G$ the set $\{n \in N : h(\delta_n) \neq e\}$ is finite, then G is n-slender.*

Proof. Let $h: \times_N \mathbb{Z} \to G$ be a homomorphism. Then, there exists n such that $h(\delta_k) = e$ for $k > n$. Suppose that $h(\times_{N\setminus\{1 \cdots n\}} \mathbb{Z}) \neq \{e\}$, and take $x \in \times_{N\setminus\{1 \cdots n\}} \mathbb{Z}$ so that $h(x) \neq e$. Let W be a word corresponding to x and $x_k = W_{N\setminus\{1 \cdots k\}}$ $(k \in N)$. Then, there exists a homomorphism $\varphi: \times_N \mathbb{Z} \to \times_N \mathbb{Z}$ such that $\varphi(\delta_k) = x_k$ $(k \in N)$ by Proposition 1.9. Now, $h \cdot \varphi(\delta_k) = h(x_k) = h(x) \neq e$ for every k, which is a contradiction.

Clearly, $\times_N \mathbb{Z}$ is not n-slender. However, $\times_N \mathbb{Z}$ is slender in the sense of [7], which is a straightforward generalization of slenderness of abelian groups. To see this, let A be an abelian subgroup of $\times_N \mathbb{Z}$. Then, A is isomorphic to $\mathbb{Z}$ or trivial by [11, Theorem 6]. Hence, $\times_N \mathbb{Z}$ is slender in the sense of [7] by Specker's theorem [16 or 6]. On the other hand, it is easy to see that every n-slender group is slender in the sense of [7].

Theorem 3.3. *An abelian group A is n-slender, if and only if A is slender.*

Proof. Let $\sigma: \times_N \mathbb{Z} \to \prod_N \mathbb{Z}$ be the canonical homomorphism. Let $h: \prod_N \mathbb{Z} \to A$ be a homomorphism for an n-slender group A. Then, there exists $n \in N$ such that $h \cdot \sigma(\times_{N\setminus\{1 \cdots n\}} \mathbb{Z}) = \{0\}$. Since $\sigma(\times_{N\setminus\{1 \cdots n\}} \mathbb{Z}) = \prod_{N\setminus\{1 \cdots n\}} \mathbb{Z}$, $h(\prod_{N\setminus\{1 \cdots n\}} \mathbb{Z}) = \{0\}$. Next, let $h: \times_N \mathbb{Z} \to A$ be a homomorphism for a slender abelian group A. Then, $h((\times_N \mathbb{Z})') = \{0\}$, where G' denotes the commutator subgroup of G. By Corollary 4.8, which we shall show in the next section, there exists no nonzero homomorphism from $\mathrm{Ker}(\sigma)/(\times_N \mathbb{Z})'$ to any slender abelian group. Hence, there exists a homomorphism $\bar{h}: \prod_N \mathbb{Z} \to A$ such that $\bar{h} = h \cdot \sigma$. Take n so that $\bar{h}(\prod_{N\setminus\{1\cdots n\}} \mathbb{Z}) = \{0\}$, then we obtain $h(\times_{N\setminus\{1\cdots n\}} \mathbb{Z}) = \{0\}$.

COROLLARY 3.4. *An n-slender group is torsion-free.*

Proof. Suppose that G is not torsion-free. Then, there exists a non-trivial finite cyclic subgroup C. Since C is abelian but not slender, C is not n-slender. Hence, G is not n-slender.

PROPOSITION 3.5. *Let S be an n-slender group and h: $\times^\sigma_{i\in I} G_i \to S$ be a homomorphism. Then, there exist a finite subset F and a homomorphism $\bar{h}$: $*_{i\in F} G_i \to S$ such that $h = \bar{h}p_F(W) = W_F$.*

Proof. First we show that $h(G_i) \neq \{e\}$ for almost all i. Suppose the contrary holds. Then there exist $i_n \in I$ $(n \in N)$ such that $h(G_{i_n}) \neq \{e\}$ for $n \in N$ and $i_m \neq i_n$ for $m \neq n$. Let $g_n \in G_{i_n}$ so that $h(g_n) \neq e$. We can naturally define a homomorphism φ: $\times_N \mathbb{Z} \to \times^\sigma_{i\in I} G_i$ such that $\varphi(\delta_n) = g_n$ $(n \in N)$ by Proposition 1.9. Then, $h \cdot \varphi(\delta_n) \neq e$ for every $n \in N$, which is a contradiction. Let $F = \{i \in I : h(G_i) \neq \{e\}\}$. Similar to the proof of Proposition 3.2, we can conclude $h(\times_{i\in I\setminus F} G_i) = \{e\}$. Since $\times_{i\in I} G_i \simeq *_{i\in F} G_i * (\times_{i\in I\setminus F} G_i)$ naturally, we obtain the conclusion.

THEOREM 3.6. *Let S_j $(j \in J)$ be n-slender groups. Then, both the restricted direct product $\prod^r_{j\in J} S_j$ and the free product $*_{j\in J} S_j$ are n-slender.*

The next corollary due to G. Higman [11, Theorem 1 with a remark on p. 80] is a fundamental result about n-slenderness.

COROLLARY 3.7 (Higman [10]). *Every free group is n-slender.*

Proof of Theorem 3.6. Let h: $\times_N \mathbb{Z} \to \prod^r_{j\in J} G_j$ be a homomorphism and $p_F: \prod^r_{j\in J} G_j \to \prod_{j\in F} G_j$ the projection for $F \Subset J$. Suppose the negation of the conclusion. By the n-slenderness of G_j, we obtain $i_n \in N$ and finite subsets J_n of J such that $i_n < i_{n+1}$, $J_n \subset J_{n+1}$, $J_n \neq J_{n+1}$, $h(\delta_{i_n}) \neq e$, $h(\delta_k) \in \prod^r_{j\in J_n} G_j$ for $1 \leqslant k \leqslant i_n$ and $p_{J_n} \cdot h(\times_{N\setminus\{1\cdots i_n\}} \mathbb{Z}) = \{e\}$. Let $\overline{W_k} = N$ with the usual ordering and $W_n(k) = \delta_{i_{n+k}}$ for $k \in N$ and $n \in N \cup \{0\}$. Let $n \in N$ be a number such that $p_j \cdot h(W_0) = e$ for $j \in \bigcup_{k\in N} J_k \setminus J_n$. By definition $p_{J_n} \cdot h(W_n) = e$ and $p_j \cdot h(\delta_{i_1} \cdots \delta_{i_n}) = e$ for $j \notin J_n$. Therefore, $p_j \cdot h(W_n) = p_j((\delta_{i_1} \cdots \delta_{i_n})^{-1} W_0) = e$ for $j \in \bigcup_{k\in N} J_k \setminus J_n$ and, consequently, $p_j \cdot h(W_n) = e$ for $j \in \bigcup_{k\in N} J_k$. By the same reasoning, $p_j \cdot h(W_{n+1}) = e$ for $j \in \bigcup_{k\in N} J_k$. Then, $p_j \cdot h(\delta_{i_{n+1}}) = p_j \cdot h(W_n \cdot (W_{n+1})^{-1}) = e$ for $j \in \bigcup_{k\in N} J_k$, which is a contradiction.

Next, to show the n-slenderness of $*_{j\in J} G_j$, let h: $\times_N \mathbb{Z} \to *_{j\in J} G_j$ be a homomorphism and σ: $*_{j\in J} G_j \to \prod^r_{j\in J} G_j$ be the canonical homomorphism.

To the contrary, suppose that $h(\delta_k) \neq e$ for infinitely many k. By n-slenderness of $\prod_{j \in J}^{r} G_j$, there exists $n \in N$ such that $\sigma \cdot h(\times_{N \setminus \{1 \cdots n\}} \mathbb{Z}) = \{e\}$. Let $\varphi\colon \times_N \mathbb{Z} \to \times_N \mathbb{Z}$ be a natural homomorphism such that $\varphi(\delta_k) = \delta_{n+k}$ according to Proposition 1.9. Then, $\sigma \cdot h \cdot \varphi(x) = e$, for $x \in \times_N \mathbb{Z}$. We claim $h \cdot \varphi(\delta_k) = e$ for almost all k, which implies the conclusion. By modifying φ, we may assume $h \cdot \varphi(\delta_k) \neq e$ for all k and $\sigma \cdot h \cdot \varphi$ is trivial. Though we can deduce a contradiction from these assumptions using Kuroš's theorem and Higman's theorem (Corollary 3.7), we present a proof which is similar to the proof of Theorem 2.1 for completeness. Remark that $l(u) \geqslant 4$ if $u \neq e$ and $\sigma(u) = e$. Let $k_1 = 1$ and $k_{n+1} = \sum_{i=1}^{n} l(h(\delta_i)) + k_n + 2$. Then, $k_j < k_{j+1}$ clearly. Let $D_j = \{s \in \mathrm{Seq} : 0 \leqslant lh(s) \leqslant j,\ 1 \leqslant s(i) \leqslant k_i \text{ for } 1 \leqslant i \leqslant j\}$ and $\overline{W}_j = D_j$ with the ordering $\prec$ and $W_j(s) = \delta_{lh(s)+1}$ for $s \in D_j$. Then, there exists a unique σ-word W such that $W_{\{1 \cdots j\}} = W_j$ for $j \in N$. Take m so that $l(h \cdot \varphi(W)) \leqslant k_m$. As in the proof of Theorem 2.1, we obtain U_s and V_s for $s \in D_m$ with $lh(s) = m$ so that $U_s = \delta_{i_1} \cdots \delta_{i_k}$, where $0 \leqslant i_1 < \cdots < i_k \leqslant m$ and $V_s \simeq Z^{k_{m+1}}$, where $\bar{Z} = \{s \in \mathrm{Seq} : 1 = s(1),\ lh(s) \geqslant 1, 1 \leqslant s(i) \leqslant k_{m+i}$ for $i \geqslant 2\}$ with $\prec$ and $Z(s) = \delta_{m+lh(s)}$. Now, $h \cdot \varphi(W) = \cdots h \cdot \varphi(u_s) \cdot (h \cdot \varphi(Z))^{k_{m+1}} \cdots$. Since H_j $(j \in J)$ are torsion-free, $h \cdot \varphi(Z)$ is a conjugate of a member of some H_m by Lemma 2.3. Since $\sigma \cdot h \cdot \varphi$ is trivial, $h \cdot \varphi(Z) = e$. By the same argument for k_{m+2} we can conclude $h \cdot \varphi(Y) = e$, where $\bar{Y} = \{s \in \mathrm{Seq} : 1 = s(1), lh(s) \geqslant 1,\ 1 \leqslant s(i) \leqslant k_{m+1+i}$ for $i \geqslant 2\}$ with $\prec$ and $Y(s) = \delta_{m+1+lh(s)}$. Then, $Z = \delta_{m+1} \cdot Y^{k_{m+2}}$ and hence $h(\delta_{m+1}) = e$, which is a contradiction.

We close this section by stating a question.

Question 3.8. Let $h\colon \times_{\omega_1} \mathbb{Z} \to \mathbb{Z}$ be a homomorphism such that $h(\delta_\alpha) = 0$ for $0 \leqslant \alpha < \omega_1$, where ω_1 is the least uncountable ordinal. Is h trivial?

It is equivalent to asking whether each homomorphism $h\colon C_{\omega_1}/(\times_{\omega_1} \mathbb{Z})^{\sigma'} \to \mathbb{Z}$ is trivial, according to the notation in Section 4.

4. Commutator Subgroups, Abelianizations, and σ-Abelianizations

Let C_I be the subgroup of $\times_I \mathbb{Z}$ consisting of x such that $p_i(x) = 0$ for all $i \in I$, where p_i is the canonical projection to the ith component. The commutator subgroup G' is the subgroup generated by all commutators $x^{-1}y^{-1}xy$ $(= [x, y])$, that is, $\langle [x, y] : x, y \in G \rangle$. Then, $G' = \langle h(C_2) : h \in \mathrm{Hom}(\mathbb{Z} * \mathbb{Z}, G) \rangle = \langle h(C_F) : h \in \mathrm{Hom}(\times_F \mathbb{Z}, G), F \Subset N \rangle$. Generalizing this in our scope, let $G^{\sigma'} = \langle h(C_N) : h \in \mathrm{Hom}(\times_N \mathbb{Z}, G) \rangle$ and $G^{\infty'} = \langle h(C_I) : h \in \mathrm{Hom}(\times_I \mathbb{Z}, G)$ for some $I \rangle$. Clearly, $G^{\sigma'}$ and $G^{\infty'}$ are normal subgroups of G. Though $G^{\infty\prime}$ also seems to be a natural subgroup of G, we have not found any interesting phenomenon about it. Therefore, we deal only with $G^{\sigma\prime}$. The abelianization of G, that is, G/G', is denoted by $\mathrm{Ab}(G)$. Similarly we define Ab^σ as $G/G^{\sigma\prime}$ and call $\mathrm{Ab}^\sigma(G)$ the σ-abelianization of G. $\mathrm{Ab}^\sigma(G)$ is a homomorphic image of $\mathrm{Ab}(G)$. To investigate $\mathrm{Ab}(G)$ and $\mathrm{Ab}^\sigma(G)$, we recall some notions for abelian groups.

An abelian group A is called complete modulo the Ulm subgroup (abbreviated by "complete mod-U"), if for any $x_n \in A$ $(n \in N)$ with $n! \mid x_{n+1} - x_n$ there exists $x \in A$ such that $n! \mid x - x_n$ for all $n \in N$. It is known that A is algebraically compact, if and only if $UU(A) = U(A)$ and A is complete mod-U [2]. A is cotorsion-free if A does not contain a non-zero cotorsion subgroup; that is, A is torsion-free, reduced, and contains no copy of the p-adic integer group $\mathbb{J}_p$ for any prime p. It is known that A is slender if and only if A is cotorsion-free and contains no copy of $\mathbb{Z}^N$. First we state some preliminary facts about this notion. Since the proofs are straightforward, we omit them.

PROPOSITION 4.1. *Any homomorphic image of a group which is complete mod-U is also complete mod-U. A direct product of groups which are complete mod-U is also complete mod-U.*

PROPOSITION 4.2. *Let A be an abelian group and H its pure subgroup. If both H and A/H are complete mod-U, then A itself is complete mod-U.*

PROPOSITION 4.3. *If an abelian group A is complete mod-U, then* $\mathrm{Hom}(A, B) = \{0\}$ *for any cotorsion-free abelian group B.*

Proof. Let $h \in \mathrm{Hom}(A, B)$. Since $\mathrm{Im}(h)$ becomes torsion-free, $U^2(\mathrm{Im}(h)) = U(\mathrm{Im}(h))$. Hence, $\mathrm{Im}(h)$ is algebraically compact by [2, Theorem 2.5]. The cotorsion-freeness of B implies $\mathrm{Im}(h) = \{0\}$.

Now, we investigate G', $G^{\sigma\prime}$, $\mathrm{Ab}(G)$, $\mathrm{Ab}^\sigma(G)$, and so on.

LEMMA 4.4. *If G is an n-slender group, then $G^{\sigma\prime} = G'$.*

Proof. Let $h\colon \times_N \mathbb{Z} \to G$ be a homomorphism. Then, there exist $F \Subset N$ and $\bar{h}\colon *_F \mathbb{Z} \to G$ such that $h = \bar{h} \cdot p_F$ by Proposition 2.4. Hence, $h(C_N) = \bar{h}(C_F) \leqslant G'$ and consequently $G^{\sigma\prime} = G'$.

THEOREM 4.5. *Let G_i $(i \in I)$ be n-slender groups. Then,* $(\times^\sigma_{i\in I} G_i)^{\sigma\prime} = \{x \in \times^\sigma_{i\in I} G_i : p_i(x) \in G_i' \text{ for all } i\}$ *and, hence,* $\mathrm{Ab}^\sigma(\times^\sigma_{i\in I} G_i) \simeq \prod^\sigma_{i\in I} \mathrm{Ab}(G_i)$ *naturally. In case of σ-products the analogous facts hold; that is,* $(\prod^\sigma_{i\in I} G_i)^{\sigma\prime} = \{x \in \prod^\sigma_{i\in I} G_i : x(i) \in G_i' \text{ for all } i\}$, *and also* $\mathrm{Ab}^\sigma(\prod^\sigma_{i\in I} G_i) \simeq \prod^\sigma_{i\in I} \mathrm{Ab}(G_i)$ *naturally.*

Proof. Since $p_j((\times^{\sigma}_{i\in I} G_i)^{\sigma'}) \leqslant G_j^{\sigma'} = G_j'$ for each j by Lemma 3.4, the one inclusion is obvious. Let $p_i(g) \in G_i'$ $(i \in I)$ for $g \in \times^{\sigma}_{i\in I} G_i$ and W be a word corresponding to g. Let $g_{i1}, ..., g_{ik_i} \in G_i$ so that the word $g_{i1} \cdots g_{ik_i}$ is $W_{\{i\}}$ for each $i \in I$. Then, $g_{i1} \cdots g_{ik_i} \in G_i'$. There exist $m_i \in N$, $h_i : *^{m_i}_{k=1} \mathbb{Z} \to G_i$ and x_{ij} $(1 \leqslant j \leqslant k_i)$ such that $h_i(x_{ij}) = g_{ij}$ and $x_{i1} \cdots x_{ik_i} \in (*^{m_i}_{k=1} \mathbb{Z})'$. Let $L = \{(i, j) : i \in I,\ \mathrm{Im}(W) \cap (G_i \backslash \{e\}) \neq \phi$ and $1 \leqslant j \leqslant m_i\}$. By $\mathbb{Z}(i, j)$, we mean the (i, j)th component of $\times_L \mathbb{Z}$. Define $h: \times_L \mathbb{Z} \to \times^{\sigma}_{i\in I} G_i$ naturally so that $h(\delta_{ij}) = h_i(\delta_j)$ for $(i, j) \in L$ according to Proposition 1.9, where δ_{ij} corresponds to 1 of $\mathbb{Z}(i, j)$. Joining all words corresponding to x_{ij}'s under the corresponding ordering of g_{ij}'s in W, we obtain a word $X \in \mathscr{W}(\mathbb{Z}(i, j) : (i, j) \in L)$ so that $h(X) = g$. Since $X_{\{(i,j) : 1 \leqslant j \leqslant k_i\}} \in (*^{m_i}_{k=1} \mathbb{Z})'$ for each i, $X \in C_L$ and, hence, $g \in (\times^{\sigma}_{i\in I} G_i)^{\sigma'}$. The second proposition follows immediately and the case of σ-products is proved analogously.

COROLLARY 4.6. $(\times_N \mathbb{Z})^{\sigma'} = C_N$.

THEOREM 4.7. *Let G_i $(i \in I)$ be n-slender groups. Then, $(\times^{\sigma}_{i\in I} G_i)^{\sigma'}/(\times^{\sigma}_{i\in I} G_i)'$ is complete modulo the Ulm subgroup.*

Proof. Let $E = \{x \in \times^{\sigma}_{i\in I} G_i : p_i(x) \in G_i'$ for all $i\}$. By Theorem 3.5 it suffices to show that $E/(\times^{\sigma}_{i\in I} G_i)'$ is complete mod-U. Since the property in question depends on countably many members only and each member is related to a σ-word, we may assume $I = N$. Let $\sigma: E \to E/(\times_{n\in N} G_n)'$ be the canonical homomorphism and $n! \mid \sigma(x_{n+1}) - \sigma(x_n)$ $(n \in N)$. We can take σ-words V_n $(n \in N)$ so that $x_{n+1} \cdot x_n^{-1} \in V_n^{n!}(\times_{n\in N} G_n)'$ and $\mathrm{Im}(V_n) \cap \bigcup^n_{k=1} G_k = \phi$. Let $B = \{s \in \mathrm{Seq} : s \neq \langle\ \rangle, 1 \leqslant s_i \leqslant i,$ for $1 \leqslant i \leqslant lh(s)\}$. Let $\overline{V_\infty} = \{(s, \alpha) : s \in B$ and $\alpha \in \overline{V_{lh(s)}}\}$, where $(s, \alpha) < (t, \beta)$ if $s \prec t$ or $s = t$ and $\alpha < \beta$ as members of $\overline{V_{lh(s)}}$ and $V_\infty(s, \alpha) = V_{lh(s)}(\alpha)$. Since $V_n \in E$ $(n \in N)$, $V_\infty \in E$. Let $\overline{U_n} = \{(s, \alpha) \in \overline{V_\infty} : lh(s) \geqslant n, s(i) = 1$ for $1 \leqslant i \leqslant n\}$ and $U_n(s, \alpha) = V_\infty(s, \alpha)$ for $(s, \alpha) \in \overline{U_n}$. Then, $V_\infty \in U_n^{n!} \cdot V_{n-1}^{(n-1)!} \cdots V_1 \cdot (\times_{n\in N} G_n)'$. Hence, $V_\infty \cdot x_1 \cdot x_n^{-1} \in U_n^{n!} \cdot x_n \cdot x_{n-1}^{-1} \cdot x_{n-1} \cdots x_2 \cdot x_1^{-1} \cdot x_1 \cdot x_n^{-1}(\times_{n\in N} G_n)' = U_n^{n!}(\times_{n\in N} G_n)'$ and, consequently, $n! \mid \sigma(V_\infty x_1) - \sigma(x_n)$ for all $n \in N$.

By Theorem 4.7 and Proposition 4.3, we obtain

COROLLARY 4.8. *Let A be a cotorsion-free abelian group. Then,* $\mathrm{Hom}(C_N, A) = \{0\}$.

COROLLARY 4.9. *Let G_i $(i \in I)$ ne n-slender groups. Then, $(\prod^{\sigma}_{i\in I} G_i)^{\sigma'}/(\prod^{\sigma}_{i\in I} G_i)'$ is complete modulo the Ulm subgroup.*

Proof. Let $\varphi: \times^{\sigma}_{i\in I} G_i \to \prod^{\sigma}_{i\in I} G_i$ and $\psi: \prod^{\sigma}_{i\in I} G_i \to \mathrm{Ab}(\prod^{\sigma}_{i\in I} G_i)$ be the canonical homomorphisms. Then, $\varphi((\times^{\sigma}_{i\in I} G_i)^{\sigma'}) = (\prod^{\sigma}_{i\in I} G_i)^{\sigma'}$ by

Theorem 4.5. For $x \in (\times^{\sigma}_{i\in I} G_i)^{\sigma\prime}$, $\psi \cdot \varphi(x) = e$ if and only if $\rho(\varphi(x)(i))$ $(i \in I)$ are bounded. Hence, $(\prod^{\sigma}_{i\in I} G_i)^{\sigma\prime}/(\prod^{\sigma}_{i\in I} G_i)'$ is a homomorphic image of $(\times^{\sigma}_{i\in I} G_i)^{\sigma\prime}/(\times^{\sigma}_{i\in I} G_i)'$. Now, the conclusion follows from Theorem 4.7 and Proposition 4.1.

To obtain further information about Ab^{σ}, we need some definitions about words.

Definition 4.10. A finite sequence of words $U_1 \cdots U_\mu$ is of n-form, if U_i $(1 \leqslant i \leqslant \mu)$ are reduced and there exist a partition $A_1 \cdots A_M, B$ of $\{1 \cdots \mu\}$ and i_k, j_k $(1 \leqslant k \leqslant m)$ such that $\{i_k, j_k : 1 \leqslant k \leqslant m\} = B$, U_{i_k} is $U_{j_k}^{-1}$ as words for each k, $U_\alpha = U_\beta$ for any $\alpha, \beta \in A_\gamma$ and $|A_\gamma| = n$ $(1 \leqslant \gamma \leqslant M)$. In addition, if the word $U_1 \cdots U_\mu$ is quasi-reduced, we say that $U_1 \cdots U_\mu$ is of canonical n-form. In case $n = 0$, we say that it is of commutator form and canonical commutator form, respectively.

Sometimes we shall confuse a sequence of words $U_1 \cdots U_k$ with a word $U_1 \cdots U_k$ for simplicity of expression.

Lemma 4.11. *Let φ: $\times_N \mathbb{Z} \to \mathrm{Ab}(\times_N \mathbb{Z})$ be the canonical homomorphism. Suppose that $\varphi(x)$ is divided by $n \in \mathbb{N}$ in $\mathrm{Ab}(\times_N \mathbb{Z})$ for $x \in \times_N \mathbb{Z}$. Then, there exists a canonical n-form $U_1 \cdots U_k$ such that $x = U_1 \cdots U_k$.*

Proof. First we describe a transformation of commutator forms corresponding to $c \in (\times_N \mathbb{Z})'$. There exists a sequence of reduced words $W_1 \cdots W_l$ of commutator form with $c = W_1 \cdots W_l$. Let $U_1 \cdots U_{2m}$ be of commutator form and $U_{i+1} \cdots U_{2m}$ is quasi-reduced. If $U_i U_{i+1} \cdots U_{2m}$ is not quasi-reduced, there exist reduced words X, V, W such that $U_i \simeq VX$, $U_{i+1} \cdots U_{2m} = X^{-1}W$, and $X^{-1}W$ is quasi-reduced. Cancelling XX^{-1} and arranging pairings, we obtain a sequence of commutator forms of length equal to or less than $2(m+1)$. Observe that the occasion "$U_i U_{i+1}$ is not quasi-reduced" happens in the process, only when $W_j = XU_i$, $W_{j+1} \cdots W_l = U_{i+1} Y$, where both XU_i and $U_{i+1} Y$ are quasi-reduced for some X and Y. Therefore, this transformation stops in finite times and we obtain a canonical commutator form which is equal to c.

Under the given condition, there exist y and $c \in (\times_N \mathbb{Z})'$ such that $x = y^n c$ and hence reduced words U and W such that $x = U^{-1} W^n U c$ and $U^{-1} W^n U$ are quasi-reduced. By a similar argument as above we obtain the conclusion.

Lemma 4.12. *There exists a pure subgroup of $\mathrm{Ab}(\times_N \mathbb{Z})$ which is also contained in $C_N/(\times_N \mathbb{Z})'$ and isomorphic to $\mathbb{Z}$.*

Proof. Let $a = e_1 \cdots e_k \cdots e_1^{-1} \cdots e_k^{-1} \cdots \in \times_N \mathbb{Z}$, where e_k is the generator of the k-component. Then, $\varphi(a) \in \varphi(C_N)$. Suppose that $\varphi(a^m)$

($m > 0$) is divided by n in $\mathrm{Ab}(\times_N \mathbb{Z})$. Then, a^m is equal to a word $U_1 \cdots U_\mu$ of canonical n-form in Definition 4.10 by Lemma 4.11. Since the reduced word of a^m is well ordered from left to right, U_α is a finite word for every $\alpha \in B$. A word of form $e_k \cdots$ for large enough k must be a part of some U_α where $\alpha \in A_\gamma$. Hence, n divides m. Now, we have shown that $\langle \varphi(a) \rangle$ is isomorphic to $\mathbb{Z}$ and a pure subgroup of $\mathrm{Ab}(\times_N \mathbb{Z})$.

THEOREM 4.13. *For a group G, $\mathrm{Ab}^\sigma(G) = G$ if and only if G is a cotorsion-free abelian group.*

Proof. If G is a cotorsion-free abelian group, $h(C_N) = 0$ for any $h \in \mathrm{Hom}(\times_N \mathbb{Z}, G)$ by Proposition 4.3 and hence $\mathrm{Ab}^\sigma(G) = G$. Now suppose that $\mathrm{Ab}^\sigma(G) = G$. Then, G is abelian. Let $\psi: \mathrm{Ab}(\times_N \mathbb{Z}) \to \mathrm{Ab}(\times_N \mathbb{Z})/U(\mathrm{Ab}(\times_N \mathbb{Z}))$ be the canonical homomorphism. Then, Lemma 4.12 implies that there exists a pure subgroup of $\mathrm{Ab}(\times_N \mathbb{Z})/U(\mathrm{Ab}(\times_N \mathbb{Z}))$ which is isomorphic to $\mathbb{Z}$ and contained in $\psi\varphi(C_N)$. Since $\psi\varphi(C_N)$ is complete mod-U by Proposition 4.1, $\psi\varphi(C_N)$ contains $\hat{\mathbb{Z}}$, that is, the $\mathbb{Z}$-adic completion of $\mathbb{Z}$, as a subgroup. The subgroup $\hat{\mathbb{Z}}$ is pure in $\mathrm{Ab}(\times_N \mathbb{Z})/U(\mathrm{Ab}(\times_N \mathbb{Z}))$ by purity of $\mathbb{Z}$ in $\mathrm{Ab}(\times_N \mathbb{Z})$. Hence, $\hat{\mathbb{Z}}$ is a summand of $\mathrm{Ab}(\times_N \mathbb{Z})/U(\mathrm{Ab}(\times_N \mathbb{Z}))$. Now, $\mathrm{Ab}^\sigma(G) = G$ implies that $h(\hat{\mathbb{Z}}) = 0$ for any $h \in \mathrm{Hom}(\hat{\mathbb{Z}}, G)$. Hence, G is cotorsion-free.

THEOREM 4.14. *Let G_i ($i \in I$) be groups where infinitely many of them are nontrivial. Then, $(\times^\sigma_{i \in I} G_i)^{\sigma\prime}/(\times^\sigma_{i \in I} G_i)'$ and hence $\mathrm{Ab}(\times^\sigma_{i \in I} G_i)$ includes a subgroup isomorphic to the direct sum of $2^{\aleph_0}$-many copies of the rational group $\mathbb{Q}$.*

To prove Theorem 4.14, some notions and lemmas are necessary.

DEFINITION 4.15. For $c \in G'$, let $\rho(c) = \min\{n : c = [x_1, y_1] \cdots [x_n, y_n]$ for $x_i, y_i \in G\}$. For $c \in \times^\sigma_{i \in I} G_i$, $\rho^*(c)$ is the minimal number m such that there exist $U_1, \ldots, U_{2m}$ of canonical commutator form with $c = U_1 \cdots U_{2m}$.

If we consider the case $G = \times_{\{0\}} G$ in the definition of ρ^* a sequence of words $U_1, \ldots, U_{2m}$ is a sequence of members of G. Therefore, $\rho^*(c)$ for $c \in G'$ depends on representations of G, which is different from the case of ρ. However, the following hold, where some G_i may be trivial.

LEMMA 4.16. *Let $G = \times^\sigma_{i \in I} G_i$. Then, $\rho(c) \leqslant \rho^*(c) - 1$ and $\rho^*(c) \leqslant 6\rho(c) - 1$ for $c \in G'$.*

Proof. Observing the role of commutators, that is, $xz^{-1}yz[yz, z^{-1}] = xy$, we can see $\rho(c) \leqslant \rho^*(c) - 1$. The second inequality follows from the proof of Lemma 4.11.

LEMMA 4.17. *Suppose that* $x \in G$ *and* $y \in H$ *satisfy* $x^2 \neq e$ *and* $y^2 \neq e$. *Then,* $\rho([x, y]^n) > n/12$ *for* $n \in N$ *in* $G * H$.

Proof. Suppose that $\rho([x, y]^n) \leqslant n/12$. Then, $\rho^*([x, y]^n) \leqslant n/2 - 1$. There exists a sequence of words $U_1 \cdots U_{2p}$ of quasi-reduced commutator form such that $p \leqslant n/2 - 1$ and $[x, y]^n = U_1 \cdots U_{2p}$. Then, one of U_i's is of form $Vy^{-1}xW$ so that V and W are nonempty and hence another one of U_i's is of form $W^{-1}x^{-1}yV^{-1}$. On the other hand, $x^{-1}y^{-1}xyx^{-1}y^{-1} \cdots xy = U_1 \cdots U_{2p}$ and $U_1 \cdots U_{2p}$ is quasi-reduced and each U_i is reduced. Since $x \neq x^{-1}$ and $y \neq y^{-1}$, it never occurs that U_i is of form $W^{-1}x^{-1}yV^{-1}$ with nonempty V and W.

Proof of Theorem 4.14. It suffices to deal with the case that $I = N$ and G_n ($n \in N$) are nontrivial groups. First we construct a subgroup which is isomorphic to $\mathbb{Q}$. Since $\times_{n \in N} G_n \simeq \times_{n \in N} (G_{2n-1} * G_{2n})$, we may assume the existence of $g_n \in G_n$ such that $g_n \neq g_n^{-1}$. Let V_n be the word $g_{2n-1}^{-1} g_{2n}^{-1} g_{2n-1} g_{2n}$ and next V_∞ and U_n ($n \in N$) the σ-words defined from V_n ($n \in N$) just in the same way as in the proof of Theorem 4.7. Then, $V_\infty \in U_n^{n!} \cdot V_{n-1}^{(n-1)!} \cdots V_1 (\times_{n \in N} G_n)' \subset (\times_{n \in N} G_n)^{\sigma\prime}$. We claim $\{\varphi(V_\infty), \varphi(U_n^a)\colon a \in \mathbb{Z},\ n \in N\}$ $(= H)$ is isomorphic to $\mathbb{Q}$, where $\varphi\colon \times_{n \in N} G_n \to \mathrm{Ab}(\times_{n \in N} G_n)$ is the canonical homomorphism. Since $U_n^n = U_{n-1} \cdot V_{n-1}^{-1} \in U_{n-1}(\times_{n \in N} G_n)'$, H is divisible and of rank 1. It suffices to show that H is torsion-free and nonzero. Suppose that $V_\infty \in (\times_{n \in N} G_n)'$. Then, $V_\infty = [x_1, y_1] \cdots [x_m, y_m]$ for some $x_i, y_i \in \times_{n \in N} G_n$, which implies $\rho([g_{2n-1}, g_{2n}]^{n!}) = \rho(p_{\{2n-1, 2n\}}(V_\infty)) \leqslant m$ for every n. This contradicts Lemma 4.17 for large enough n. By a similar argument we can see that H is torsion-free. To get the conclusion of the lemma, we modify the above construction. There exist $X_\alpha \subset N$ ($\alpha < 2^{\aleph_0}$) such that each X_α is infinite and $X_\alpha \cap X_\beta$ is finite for distinct α, β. Let $k_{n\alpha}$ ($n \in N$) be an enumeration of X_α without repetition. Let $V_{\infty\alpha}$ and $U_{n\alpha}$ be the σ-words obtained by replacing n by $k_{n\alpha}$ in the above construction. Then, we obtain subgroups H_α ($\alpha < 2^{\aleph_0}$) of $(\times_{n \in N} G_n)^{\sigma\prime}/(\times_{n \in N} G_n)'$ which are linearly independent and isomorphic to $\mathbb{Q}$.

Since a theorem analogous to Theorems 4.14 holds for σ-products of free products, we prove it in the remaining part of this section. We need a lemma which is a version of Lemma 4.17. As is well known, the commutator subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$, that is, the infinite dihedral group, consists of all commutators, where $\mathbb{Z}_2$ is the group of order 2. Except in this case we obtain the following

LEMMA 4.18. *Let* G *and* H *be nontrivial groups at least one of which is not isomorphic to* $\mathbb{Z}_2$. *Then, there exists* $c \in (G * H)'$ *such that* $\rho^*(c^m) > (m-1)/2$ *and consequently* $\rho(c^m) > (m+1)/12$ *for* $m \in N$.

Proof. We assume that G is not isomorphic to $\mathbb{Z}_2$.

Case 1. There exists $g \in G$ such that $g \neq g^{-1}$. Take $h \in H$ with $h \neq e$ and let $c = g^{-1}hghghghg^{-1}h^{-1}g^{-1}h^{-1}gh^{-1}g^{-1}h^{-1}$. Then, $c^{-1} = hghg^{-1}hghgh^{-1}g^{-1}h^{-1}g^{-1}h^{-1}g^{-1}h^{-1}g$. We only remark on the ordering of g and g^{-1} and the fact $h, h^{-1} \neq e$. Then, we can conclude $\rho^*(c^m) > (m-1)/2$ by a similar reasoning to the proof of Lemma 4.17. Hence, $\rho(c^m) > (m+1)/12$ by Lemma 4.16.

Case 2. Otherwise. Then, $g^2 = e$ for every $g \in G$. Take distinct $g_1, g_2 \in G$ with $g_1, g_2 \neq e$ and $h \in H$ with $h \neq e$ and let $c = hg_1g_2hg_1hg_2h^{-1}g_1g_2h^{-1}g_1h^{-1}g_2$. Since $g_1g_2 \neq g_1$ and $g_1g_2 \neq g_2$, we conclude that $\rho^*(c^m) > (m-1)/2$ and $\rho(c^m) > (m+1)/12$ as before.

THEOREM 4.19. *Let G_i and H_i be nontrivial groups at least one of which is not isomorphic to $\mathbb{Z}_2$ and I an infinite index set, then $(\prod^\sigma_{i \in I} G_i * H_i)^{\sigma\prime}/(\prod^\sigma_{i \in I} G_i * H_i)'$ and consequently* $\mathrm{Ab}(\prod^\sigma_{i \in I} G_i * H_i)$ *include a subgroup isomorphic to the direct sum of $2^{\aleph_0}$-many copies of the rational group $\mathbb{Q}$.*

Proof. It is enough to prove this in case $I = N$. Let $\varphi\colon \prod_{n \in N} G_n * H_n \to \mathrm{Ab}(\prod_{n \in N} G_n * H_n)$ be the canonical homomorphism. Take $c_n \in (G_n * H_n)'$ $(n \in N)$ so that $\rho(c_n^m) > (m+1)/12$ for $m \in N$ by Lemma 4.18. Define $x_m \in \prod_{n \in N} G_n * H_n$ $(m \in N)$ by: $x_m(n) = e$ for $n < m$ and $x_m(n) = c_n^{n!/m!}$ for $n \geqslant m$. Let $H = \{\varphi(x_m^a) : m \in N, a \in \mathbb{Z}\}$. Then, H is isomorphic to $\mathbb{Q}$. The rest of the proof is similar to that of Theorem 4.14.

We remark that $(\prod_N (\mathbb{Z}_2 * \mathbb{Z}_2))' = \prod_N (\mathbb{Z}_2 * \mathbb{Z}_2)'$ and hence $\mathrm{Ab}(\prod_N (\mathbb{Z}_2 * \mathbb{Z}_2)) \simeq \prod_N (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ canonically.

Question 4.20. Are $\mathrm{Ab}(\times_N \mathbb{Z})$ and $\mathrm{Ab}(\prod_N (\mathbb{Z} * \mathbb{Z}))$ torsion-free?

It is equivalent to ask whether $(\times_N \mathbb{Z})^{\sigma\prime}/(\times_N \mathbb{Z})'$ and $(\prod_N (\mathbb{Z} * \mathbb{Z}))^{\sigma\prime}/(\prod_N (\mathbb{Z} * \mathbb{Z}))'$ are torsion-free or not. Especially, $(\prod_N (\mathbb{Z} * \mathbb{Z}))^{\sigma\prime}/(\prod_N (\mathbb{Z} * \mathbb{Z}))'$ is not torsion-free, if and only if there exist $m, M \in N$ such that $\sup\{\rho(c) : c \in (\mathbb{Z} * \mathbb{Z})', \rho(c^m) \leqslant M\} = \infty$. If the answers to these questions are affirmative, $\mathrm{Ab}(\times_N \mathbb{Z})$ and $\mathrm{Ab}(\prod_N (\mathbb{Z} * \mathbb{Z}))$ have summands isomorphic to $\mathbb{Z}^N$ and $(\mathbb{Z} \oplus \mathbb{Z})^N$, respectively, by Theorem 4.7.

APPENDIX

Here, we state applications to algebraic topology, which are the background of the context. Topological spaces in this appendix are always Hausdorff. Undefined notions about algebraic topology are standard and can be found in [12, 15]. Let (X_i, x_i) be pointed spaces such that $X_i \cap X_j = \phi$ for $i \neq j$. There are two typical ways of attaching spaces (X_i, x_i)

under the identification of all x_i $(=x^*)$. The underlying set of the two spaces $\bigvee_{i\in I}(X_i, x_i)$ and $\tilde{\bigvee}_{i\in I}(X_i, x_i)$ are $\{x^*\} \cup \bigcup_{i\in I} X_i \backslash \{x_i\}$. It suffices to define the open neighborhood of x^*. Let U be a subset of $\{x^*\} \cup \bigcup_{i\in I} X_i \backslash \{x_i\}$ containing x^*. U is open in $\bigvee_{i\in I}(X_i, x_i)$ if $U \cap X_i$ is open in each X_i, while U is open in $\tilde{\bigvee}_{i\in I}(X_i, x_i)$ if U is open in $\bigvee_{i\in I}(X_i, x_i)$ and $(U\backslash\{x^*\}) \cap X_i = X_i \backslash \{x_i\}$ for almost all i. If each X_i is locally simply connected at x_i and also first countable at x_i, then $\pi_1(\bigvee_{i\in I}(X_i, x_i)) \simeq *_{i\in I} \pi_1(X_i)$. Here, the first countability is essential even if I is finite [3, 4]. On the other hand, we have the following, which was a theorem of H. B. Griffiths [9] and J. W. Morgan and I. Morrison [14] have completed its proof.

THEOREM A.1 (H. B. Griffiths, J. W. Morgan and I. Morrison). *Let X_i be locally simply connected at x_i and also have countable basic neighborhoods of x_i for each $i \in I$. Then, $\pi_1(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i\in I} \pi_1(X_i, x_i)$.*

Originally this was proved in case I is countable, but it is not hard to see that this also holds for arbitrary I, which we shall explain in the sequel. In the introduction of [14], they stated that the proof contains a noneffective construction of a homotopy. Though we do not insist that the following proof is effective, it seems that it is more direct. Since the proof will clarify the meaning of equivalence of infinitary words, we outline the proof and present a direct construction of a homotopy.

For a pointed space (X, x) a loop f in (X, x) is a continuous map from a closed interval $[a, b]$ (where $a<b$) to a space with $f(a)=f(b)=x$. Two loops f and g in (X, x) with their domain $[a, b]$ are briefly said to be homotopic, if there exists a homotopy from f to g which is constant relative to a and b. When we do not mention the domain of a loop, the domain is always $[0, 1]$. For an interval I, $\dot{I}$ is the set of end points of I. For a loop f in $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ there exist at most countable pairwise disjoint open subintervals (a_n, b_n) $(n \in M)$ of $[0, 1]$ such that $\bigcup_{n\in M}(a_n, b_n) = f^{-1}(\tilde{\bigvee}_{i\in I}(X_i, x_i)\backslash\{x^*\})$. Each loop $f \mid [a_n, b_n]$ lies in some X_i and for each i almost all loops $f \mid [a_n, b_n]$ in (X_i, x_i) are homotopic to the constant map, since X_i is locally simply connected at x_i. Hence, we can obtain a σ-word $W^f \in \mathcal{W}(\pi_1(X_i, x_i) : i \in I)$ naturally using the ordering of (a_n, b_n)'s. If f is homotopic to the constant map, it is easy to see that $(W^f)_F = e$ for any $F \Subset J$. Hence, we can define a natural homomorphism $\psi\colon \pi_1(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \to \times^{\sigma}_{i\in I} \pi_1(X_i, x_i)$ by: $\pi_F \cdot \psi([f]) = (W^f)_F$ for $F \Subset J$, where $[f]$ is the member of $\pi_1(\tilde{\bigvee}_{i\in I}(X_i, x_i))$ corresponding to f. It is also easy to see that ψ is surjective. To see the injectivity of ψ, some notion is necessary. A loop f in (X, x) is proper, if f satisfies the following: Let (a_n, b_n) $(n \in M)$ be pairwise disjoint open intervals such that $\bigcup_{n\in M}(a_n, b_n) = f^{-1}(X\backslash\{f(0)\})$. Then, if $f \mid [a_n, b_n]$ is homotopic to the

constant loop, $f \mid [a_n, b_n]$ itself is constant. The next lemma is the only part where we use the first countability.

LEMMA A.2 (Essentially in [8, 1.2]. *Let X be locally simply connected at x which has countable neighborhood bases. Let f be a loop in $((X, x) \vee (Y, y), x^*)$ such that $f(a_n) = f(b_n) = x^*$ for $n \in N$, $f([a_n, b_n]) \subset X$, and $f \mid [a_n, b_n]$ is homotopic to the constant loop, where (a_n, b_n) $(n \in N)$ are pairwise disjoint open subintervals of $[0, 1]$. Then, there exists a continuous map H: $[0, 1] \times [0, 1) \to (X, x) \vee (Y, y)$ with the following*:

(1) $H(1, t) = f(t)$ *for* $t \in [0, 1]$;

(2) $H(s, 0) = H(s, 1) = H(s, a_n) = H(s, b_n) = x$ *for* $s \in [0, 1]$ *and* $n \in N$;

(3) $H(s, t) \in X$ *for* $s \in [0, 1]$ *and* $t \in \bigcup_{n \in N} [a_n, b_n]$;

(4) $H(0, t) = x$ *for* $t \in \bigcup_{n \in N} [a_n, b_n]$.

Since this is not so hard to prove if we use the two given local properties, we omit the proof. Since the image of any loop f in $(\tilde{\bigvee}_{i \in I} (X_i, x_i), x^*)$ is included by $\tilde{\bigvee}_{i \in C} (X_i, x_i)$ for some countable $C \subset I$, by iterating use of this lemma we obtain

LEMMA A.3 [14, Lemma 4.2]. *Under the same conditions as in Theorem* A.1, *any loop f in $(\tilde{\bigvee}_{i \in I} (X_i, x_i), x^*)$ is homotopic to some proper loop.*

Proof of Theorem A.1. Let f be a loop in $(\tilde{\bigvee}_{i \in I} (X_i, x_i), x^*)$ with $W^f = e$. Since there exists a countable subset C of I such that $\mathrm{Im}(f) \subset \tilde{\bigvee}_{i \in C} (X_i, x_i)$, it suffices to deal the case $I = N$. By Lemma A.3, we may assume that f is a proper loop. Now, we construct a homotopy H from f to the constant loop. In the kth step, we define H on subrectangles of $[0, 1] \times [0, 1]$ which makes loops in (X_k, x^*) homotopic to the constant loop expecting loops in $\tilde{\bigvee}_{n > k} X_n$ will be made homotopic to the constant loop in a suitable way in future.

(Step 1) Let $H(t, 1) = f(t)$ and $H(t, 0) = x^*$ for $0 \leqslant t \leqslant 1$. Let $W^f = W_1 \cdots W_{n_1}$, where $W_i \in \mathscr{W}(G_1)$ or $W_i \in \mathscr{W}(G_n : n \geqslant 2)$ for $1 \leqslant i \leqslant n_1$ and $W_i \in \mathscr{W}(G_1)$ if and only if $W_{i+1} \in \mathscr{W}(G_j : j \geqslant 2)$ for $1 \leqslant i \leqslant n_1 - 1$.

(Substep 1) We can correspond a closed interval I_i to each W_i so that $W_i = W^{f \mid I_i}$ for $1 \leqslant i \leqslant n_1$, $\bigcup_{i=1}^{n_1} I_i = [0, 1]$, and the right end of I_i is the left end of I_{i+1} for $1 \leqslant i \leqslant n_1 - 1$. We claim that $W_i = e$ for some $1 \leqslant i \leqslant n_1$. Suppose not. There exists $F \Subset N$ such that $p_F(W_i) \neq e$ for every $1 \leqslant i \leqslant n_1$. Then, $p_F(W^f) \neq e$, which is a contradiction. We choose one W_i with $W_i = e$. Let $H(s, t) = f(s)$ for $(s, t) \in \bigcup_{j \neq 1} I_j \times [1/2, 1]$.

In case $W_i \in \mathscr{W}(G_1 : n \geqslant 2)$, $f \mid I_i$ is homotopic to the constant loop in

X_1. Let $H \mid I_i \times [\frac{1}{2}, 1]$ be a continuous map such that $H(s, \frac{1}{2}) = x^*$ for $s \in I_i$ and $H(s, t) = x^*$ for $s \in \dot{I}_i$ and $t \in [\frac{1}{2}, 1]$. In case $W_i \in \mathscr{W}(G_n : n \geqslant 2)$ we do not define H on $(I_i \backslash \dot{I}_i) \times (\frac{1}{2}, 1)$ in this step, but we let $H(s, \frac{1}{2}) = x^*$ for $s \in I_i$. Next, we reform the word W^f to $W_1 \cdots V \cdots W_{n_1}$ by eliminating W_i, where $V = W_{i-1} W_{i+1}$. Then, $W_1 \cdots V \cdots W_{n_1} = e$ and members of $\mathscr{W}(G_1)$ and $\mathscr{W}(G_n : n \geqslant 2)$ are neighboring in $W_1, ..., V, ..., W_{n_1}$.

(Substep $k+1$) In the substep k, $H(s, 1/2^k)$ ($s \in [0, 1]$) have been defined and there is a corresponding word reformed from W^f. By the same reasoning as in Substep 1, one of the words equals e as a member of the group, of course. We perform the work as in Substep 1. The substeps would finish in at most n_1-steps. If they finish in the k-step, then $H(t, 1/2^k)$ $(0 \leqslant t \leqslant 1)$ have been defined and equal to x^*. Let $H([0, 1] \times [0, 1/2^k]) = x^*$.

(Step k) After the $(k-1)$-step, there possibly exist finitely many subrectangles of $[0, 1] \times [0, 1]$ on which H has not been defined. Their forms are $[a, b] \times (\sum_{i=1}^{m-1} s_i/2^i + 1/2^m, \sum_{i=1}^{m-1} s_i/2^i + 1/2^{m-1})$, where $s_i = 0$ or 1 and $m \leqslant \sum_{i=1}^{k} n_i$. H has been defined on the upper side of a rectangle and it corresponds to a word in $\mathscr{W}(\pi_n(X_n, x_n) : n \geqslant k)$. H maps the lower side to x^*. In each rectangle, we work as in Step 1, as if the rectangle were $[0, 1] \times [0, 1]$. Note that the values of H which we define in this step are in $\tilde{\bigvee}_{n>k} (X_n, x_n)$, because the loops in question are in $\tilde{\bigvee}_{n \geqslant k} (X_n, x_n)$.

Let $H(t, u) = x^*$, if $H(t, u)$ has not been defined in any step. Now, the continuity of H is clear and the proof of Theorem A.1 is complete.

Next we state a characterization of n-slender groups using π_1-groups. Some preliminaries and definitions are necessary to state it.

A continuous map $f: X \to Y$ with $f(x) = y$ naturally induces a homomorphism $f_*: \pi_1(X, x) \to \pi_1(Y, y)$. A homomorphism $h: \pi_1(X, x) \to \pi_1(Y, y)$ is spatial with respect to pointed spaces (X, x) and (Y, y), if there exists a continuous map $f: X \to Y$ with $f(x) = y$ such that $f_* = h$. Denote the circle with a base point by $(\mathbb{S}^1, b)$ and let $(\mathbb{S}^1_n, b_n)$ $(n \in N)$ be copies of it. Then, $\tilde{\bigvee}_{n \in N} (\mathbb{S}^1_n, b_n)(= (\mathbb{H}, b^*))$ is the so-called Hawaiian earring.

THEOREM A.4. *For a group G the following are equivalent*:

(1) *G is n-slender*;

(2) *Let X_i $(i \in I)$ be 2-simplicial complexes with $x_i \in X_i$. If $\pi_1(Y, y) \simeq G$ for an arbitrary pointed space (Y, y), then any homomorphism $h: \pi_1(\tilde{\bigvee}_{i \in I} (X_i, x_i), x^*) \to \pi_1(Y, y)$ is spatial with respect to $(\tilde{\bigvee}_{i \in I} (X_i, x_i), x^*)$ and (Y, y)*;

(3) *If $\pi_1(Y, y) \simeq G$ for a pointed space (Y, y), then any homomorphism $h: \pi_1(\mathbb{H}, b^*) \to \pi_1(Y, y)$ is spatial with respect to $(\mathbb{H}, b^*)$ and (Y, y).*

Proof. It suffices to show the implications $(1) \to (2)$ and $(3) \to (1)$.

$(1) \to (2)$ By Theorem A.1, $\pi_1(\tilde{\bigvee}_{i \in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i \in I} \pi_1(X_i, x_i)$ naturally. Therefore, there exist $E \Subset I$ and $\bar{h}: \pi_1(X_i, x_i) \to \pi_1(Y, y)$ such that $h = \bar{h} \cdot p_E$ by Proposition 3.5. Since $X_i (i \in I)$ are 2-simplicial complexes, a standard method shows that any homomorphism from $\pi_1(X_i, x_i)$ to $\pi_1(Y, y)$ is spatial. Hence, there exist continuous maps $f_i: X_i \to Y$ $(i \in E)$ such that $f_i(x_i) = y$ and $(f_i)_* = h \mid \pi_1(X_i, x_i)$. Define a continuous map $f: \tilde{\bigvee}_{i \in I}(X_i, x_i) \to Y$ by: $f \mid X_i = f_i$ for $i \in E$ and $f(\tilde{\bigvee}_{i \in I \setminus F}(X_i, x_i)) = \{y\}$. Then, $f_* = h$.

$(3) \to (1)$ Let $\times_N \mathbb{Z} \to G$ be a homomorphism. Then, there exists a simplicial complex Y with y such that $\pi_1(Y, y) \simeq G$, for example, the Eilenberg–Maclane complex $K(1, G)$. Identify $\times_N \mathbb{Z}$ with $\pi_1(\mathbb{H}, b^*)$, then there exists a continuous map $f: \mathbb{H} \to Y$ such that $f(b^*) = y$. Since Y is locally contractible, there exists $n \in N$ such that $f(\tilde{\bigvee}_{m \geqslant n}(\mathbb{S}^1_m, b_m))$ contained in some contractible neighborhood of y. On the other hand, $\mathbb{H} = \bigvee_{m < n}(\mathbb{S}^1_m, b_m) \vee \tilde{\bigvee}_{m \geqslant n}(\mathbb{S}^1_m, b_m)$ and $\pi_1(\mathbb{H}, b^*) \simeq *_{m<n} \pi_1(\mathbb{S}^1_m, b_m) * \times_{m \geqslant n} \pi_1(\mathbb{S}^1_m, b_m)$ naturally. Therefore,

$$h(\times_{N \setminus \{1 \cdots m\}} \mathbb{Z}) = f_*(\times_{m \geqslant n} \pi_1(\mathbb{S}^1_m, b_m)) = \{e\}.$$

As is well known, the first integral singular homology group $H_1(X)$ is isomorphic to $\mathrm{Ab}(\pi_1(X, x))$ for a path-connected pointed space (X, x) [14]. For certain spaces we can interpret $\mathrm{Ab}^{\sigma}(G)$ naturally. Let X_i $(i \in I)$ be simplicial complexes, or ANRs more generally, such that $\pi_1(X_i, x_i)$ $(i \in I)$ are n-slender. Then, $\pi_1(\tilde{\bigvee}_{i \in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i \in I} \pi_1(X_i, x_i)$. As we have shown in Section 3, $H_1(\tilde{\bigvee}_{i \in I}(X_i, x_i))$ becomes a rather complicated group for an infinite I, even if X_i $(i \in I)$ are copies of $\mathbb{S}^1$. On the other hand, the factor $H^T_1(\tilde{\bigvee}_{i \in I}(X_i, x_i))$ of $H_1(\tilde{\bigvee}_{i \in I}(X_i, x_i))$, introduced in [5], is naturally isomorphic to $\prod^{\sigma}_{i \in I} H^T_i(X_i)$ for path-connected spaces (X_i, x_i) by [5, Theorem 4.6]. Therefore, $H^T_1(\tilde{\bigvee}_{i \in I}(X_i, x_i))$ is isomorphic to $\mathrm{Ab}^{\sigma}(\pi_1(\tilde{\bigvee}_{i \in I}(X_i, x_i), x^*))$. We explain the situation a little.

$H_1(\tilde{\bigvee}_{i \in I}(X_i, x_i))$ consists of loops modulo the image of the boundary map, $\mathrm{Im}(\partial_2)$. A loop f with base point x^* represents an element of $\pi_1(\tilde{\bigvee}_{i \in I}(X_i, x_i), x^*)'$ if and only if f belongs to $\mathrm{Im}(\partial_2)$. H^T_1 is defined as H_1 replacing $\mathrm{Im}(\partial_2)$ by $\overline{\mathrm{Im}(\partial_2)}$, where the topological closure is taken under the topology of a free topological abelian group. (See [5] for precise definition.) Now, f represents an element of $\pi_1(\tilde{\bigvee}_{i \in I}(X_i, x_i), x^*)^{\sigma'}$ if and only if f belongs to $\overline{\mathrm{Im}(\partial_2)}$. Hence, $\overline{\mathrm{Im}(\partial_2)}/\mathrm{Im}(\partial_2)$ is complete modulo the Ulm subgroup by Theorem 4.7 in such a case.

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